Phys 410
Fall 2014
Lecture #10 Summary
2 October, 2014

Making Newton's second law work in a rotating reference frame is a challenge. Consider a rigid body undergoing pure rotational motion on an axis through a fixed point inside the object. We found that the linear velocity of a particle at location $\vec{r}$ inside or on the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$.

We then calculated the relationship between the time-derivative of a vector $\vec{Q}$ as seen in an inertial reference frame S_0, to the derivative of the same vector seen in the rotating reference frame S. We assume that the two reference frames have the same origin, but frame S is rotating about an arbitrary axis $\widehat{\Omega}$ through the origin at a rate Ω. The time-derivatives are related as $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$. This equation says that the time derivative of the vector as witnessed in the inertial reference frame consists of any change in its magnitude or direction as seen in the non-inertial reference frame, plus the change brought about by the fact that the vector $\vec{Q}$ is embedded in a rotating rigid body.

Newton's second law can now be written for an observer in a rotating reference frame as $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$. The two "inertial forces" on the right are called the Coriolis force and the centrifugal force, respectively.

We considered the centrifugal ("center-fleeing") force for a stationary observer on the surface of the earth. This force has a direction that is directly away from the axis of rotation of the earth and can be written as $\vec{F}_{CF} = m\Omega^2 r \sin\theta\, \hat{\rho}$, where r is the distance from the center of the earth, θ is the polar angle of the location on the surface (also known as the co-latitude) and $\hat{\rho}$ is the radial unit vector from cylindrical coordinates. This force has a maximum magnitude near the equator, but goes to zero at the poles. The centrifugal force modifies the free-fall direction. It creates a new effective gravitational acceleration vector of $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$, where $\vec{g}_0$ is the bare Newtonian gravity acceleration vector that points directly to the center of the earth, and R is the radius of the earth. The radial component of this vector is $g_{rad} = g_0 - \Omega^2 R \sin^2\theta$, showing that things weigh a bit less at the equator than at the north/south pole. The effect is small, only about 0.3%. The tangential component of $\vec{g}$ is $g_{tang} = \Omega^2 R \sin\theta \cos\theta$, with a maximum value at 45° latitude. This component produces a 0.1° tilt of $\vec{g}$ with respect to the direction of $\vec{g}_0$.

The Coriolis force $\vec{F}_{Cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$ depends on the state of motion of the object. In fact it resembles the force on a charged particle in a magnetic field. The 'charge' is $2m$ and the 'magnetic field' is the angular velocity vector $\vec{\Omega}$. The particle will be deflected as it travels through this 'field'. In the northern hemisphere the deflection is to the right, while in the southern hemisphere it is in the opposite direction because $\vec{\Omega}$ has a substantial component into the ground (hence the phrase 'down under'). The magnitude of the Coriolis force for an object on the surface of the earth moving at 50 m/s is quite small, resulting in an acceleration of at most 0.007 m/s^2. The Coriolis force is significant for objects with large mass (air masses, hurricanes, etc.), or for objects moving quickly (artillery shells and ICBMs).

We considered the motion of the Foucault pendulum. The demonstration showed that the pendulum moves in a fixed plane, as seen from an inertial reference frame. An inertial observer sees that the plane of oscillation is fixed and that the forces acting on the bob create no torque that will cause the plane of oscillation to change. However, in a rotating reference frame, the pendulum appears to move in a series of planes that rotate clockwise, as seen from above (in the northern hemisphere). The pendulum is made of a light wire of length L supporting a bob of mass m. The equation of motion of the bob as seen in the non-inertial frame is $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$, where the net force identified from an inertial reference frame is the vector sum of tension in the wire and gravity: $\vec{F}_{net} = \vec{T} + m\vec{g}_0$. This is the bare gravity force that points toward the center of the earth. Last time we saw that bare gravity can be combined with the centrifugal force and re-named effective gravity: $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$. We designate "up" or the +z-direction to be the direction away from $\vec{g}$, and y to be the "north" direction, and x to be the "east" direction. In this way, the angular velocity vector for the earth $\vec{\Omega}$ points in the y-z plane at an angle θ with respect to the "up" (z) direction.

The z-motion of the bob is fairly simple, essentially reducing to the statement that $T \cong mg$. The tension in the horizontal xy-plane is $T_x = -mgx/L$, and $T_y = -mgy/L$. The Coriolis force is found from the cross product $2m\dot{\vec{r}} \times \vec{\Omega}$. We write $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ and $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$. After carrying out the cross product and putting the results into the equation of motion, broken down into components, we get: $m\ddot{x} = -\frac{mgx}{L} + 0 + 2m(\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta)$, and $m\ddot{y} = -\frac{mgy}{L} + 0 - 2m\dot{x}\Omega\cos\theta$. We shall drop the $\dot{z}$ term in the x-equation because it is the product of two small velocities, define the constants $\omega_0^2 \equiv g/L$, and $\Omega_z \equiv \Omega\cos\theta$, to get two coupled equations of motion:

$$\ddot{x} - 2\dot{y}\Omega_z + \omega_0^2 x = 0$$

$$\ddot{y} + 2\dot{x}\Omega_z + \omega_0^2 y = 0$$

The first and third terms alone would give un-coupled simple harmonic motion in the xy-plane. The coupling terms look like a form of dissipation (of the form $F_{dis} = -bv$) but in fact they represent a coupling of energy from one direction of motion to the other. The energy in the oscillations sloshes back and forth between x and y.

These equations can be combined in a manner similar to the equations for motion of a charged particle in a magnetic field. Take the first equation plus "i" times the second equation, and define the new dependent complex variable $\eta(t) \equiv x(t) + iy(t)$ to get a single equation: $\ddot{\eta} + i2\dot{\eta}\Omega_z + \omega_0^2\eta = 0$. Trying a solution of the form $\eta(t) = e^{-i\alpha t}$, we get an auxiliary equation with solutions $\alpha = \Omega_z \pm \sqrt{\omega_0^2 + \omega\Omega_z^2}$. Using the fact that the pendulum oscillates many times compared to the rotation period of the Earth (i.e. $\omega_0 \gg \Omega_z$) we come to the solution $\eta(t) = e^{-i\Omega_z t}\left(C_1 e^{-i\omega_0 t} + C_2 e^{+i\omega_0 t}\right)$. To supply initial conditions, consider pulling the pendulum bob to a displacement A in the east (x) direction ($y = 0$) and release it from rest. In this case one finds $C_1 = C_2 = A/2$, and the solution is $\eta(t) = Ae^{-i\Omega_z t}\cos(\omega_0 t)$. Taking the real and imaginary parts to get the actual equations of motion in real space gives $x(t) = A\cos(\Omega_z t)\cos(\omega_0 t)$ and $y(t) = -A\sin(\Omega_z t)\cos(\omega_0 t)$. The pendulum swings back and forth on a short time scale, described by the factor of $\cos(\omega_0 t)$. On longer time scales, the plane of oscillation rotates, as described by the factors of $\cos(\Omega_z t)$ and $-\sin(\Omega_z t)$, with $\omega_0 \gg \Omega_z$. This slow rotation of the plane of oscillation occurs at a frequency that depends on your (co-)latitude on the Earth $\Omega_z \equiv \Omega\cos\theta$, where the rotation frequency of the Earth is $\Omega = 7 \times 10^{-5}$ Rads/s.